MINIMAL VOLUME INVARIANTS, TOPOLOGICAL SPHERE THEOREMS AND BIORTHOGONAL CURVATURE ON 4-MANIFOLDS

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ABSTRACT. We provide new estimates to Yamabe minimal volume and mixed minimal volume involving some topological invariants on four-dimensional compact manifolds. Moreover, we obtain topological sphere theorems for compact submanifolds of spheres and Euclidean spaces, provided that the full norm of the second fundamental form is bounded. In addition, we obtain topological obstructions to the existence of Einstein metrics on four-dimensional manifolds.

1. INTRODUCTION

A distinguished problem in differential geometry is to study the relationships between curvature and topology on four-dimensional compact manifolds. This is because dimension four enjoys a privileged status; many relevant facts may be found in [5] and [41]. In 1980s, Donaldson used the Yang-Mills equations to study the differential topology of four-dimensional manifolds. Using Freedman's work, he was able to prove that there exist many compact topological four-dimensional manifolds which have no smooth structure. Moreover, he proved that there exist many pairs of compact simply connected smooth four-dimensional manifolds which are homeomorphic but not diffeomorphic. In the fall of 1994, Witten and Seiberg proposed a couple of equations which provide some results of Donaldson's theory in a far simpler way than had been thought possible. In other words, E. Witten discovered a family of invariants on four-dimensional manifolds, now called the Seiberg-Witten invariants, obtained by a counting solutions of a non-linear Dirac equation. Witten [49] observed that a compact four-manifold with positive scalar curvature must have vanishing Seiberg-Witten invariants. Thus there is an obstruction to the existence of metrics of positive scalar curvature which depends on the differentiable structure of the four-manifold, not just its topological type. The Seiberg-Witten invariants have become one of the standard tools in studying the differential topology of four-dimensional manifolds. We refer to [15, 41, 49] for the definitions and basic properties of Seiberg-Witten invariants. These above comments tell us, in particular, that dimension four is really special.

1.1. **Minimal Volume Invariants.** In order to set up the notation, M^n will denote an n-dimensional compact oriented smooth manifold with scalar curvature s_g,

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or simply s, sectional curvature K and $\mathscr{M}$ denotes the set of smooth Riemannian structures on M^n. We further denote by $\chi(M)$ the Euler characteristic of M^n. While the signature of M^n is denoted by $\tau(M)$, which is, in module, a topological invariant. We consider all complete Riemannian structures $g \in \mathscr{M}$ whose sectional curvatures satisfy $|K(g)| \leq 1$. With these settings, Gromov [17], in his seminal paper on bounded cohomology, introduced the concept of minimal volume. More precisely, the *minimal volume* of M^n is defined by

$$\mathrm{Min\,Vol}(M) = \inf_{|K(g)|\leq 1} Vol(M,g). \tag{1.1}$$

This concept plays an important role in geometric topology. It is closely related with others important invariants as, for instance, *minimal entropy* $\mathrm{h(M)}$ and *simplicial volume* $\|M\|$. Paternain and Petean [38] proved that the minimal volume, on a compact manifold M^n, satisfies the following chain of inequalities

$$c(n)\|M\| \leq [\mathrm{h(M)}]^n \leq (n-1)^n \mathrm{Min\,Vol(M)}, \tag{1.2}$$

where $c(n)$ is a positive constant; for more details, we refer the reader to [24, 25] and [38]. One should be emphasized that some authors have been studied others minimal volumes invariants in the same context; see [31, 45] and [46]. Among them, we detach the next ones: *Gromov minimal volume*, which is defined by

$$\mathrm{Vol}_K(M) = \inf\{Vol(M,g); K_g \geq -1\}. \tag{1.3}$$

Also, *the Yamabe minimal volume* which is defined by

$$\mathrm{Vol}_s(M) = \inf\left\{Vol(M,g); \frac{s_g}{n(n-1)} \geq -1\right\}. \tag{1.4}$$

It measures how much the negative scalar curvature is inevitable on a compact manifold. In fact, if a compact manifold M^n admits a metric with nonnegative scalar curvature, then $\mathrm{Vol}_s(M) = 0$. In general, the computation of these invariants is not easy. However, Petean [40] showed brightly that any compact simply connected manifold M^n of dimension $n \geq 5$ has $\mathrm{Vol}_s(M) = 0$. More precisely, it collapses with scalar curvature bounded from below. While LeBrun [31] defined a new type of minimal volume called *mixed minimal volume*, which is given by

$$\mathrm{Vol}_{K,s}(M) = \inf\left\{Vol(M,g); \frac{1}{2}\left(K_g + \frac{s_g}{n(n-1)}\right) \geq -1\right\}. \tag{1.5}$$

Another interesting mixed minimal volumes were studied by C. Sung (cf. [45] and [46]).

We recall that a compact complex surface M^4 is said of *general type* if the Kodaira dimension of M^4 is equal to 2; for details see [29]. Lebrun [28] showed that a compact complex surface M^4 with first Betti number $b_1(M)$ even is of general type if and only if the Yamabe invariant $Y(M)$ is negative. However, in dimensions ≥ 4, the Yamabe invariant alone is weak to control the topology of a given manifold. For this reason, it is expected one additional condition. We highlight that the hypothesis that $b_1(M)$ even is equivalent to requiring that the complex surface M^4 admit a Kähler metric. Moreover, from Seiberg-Witten theory any two diffeomorphic complex algebraic surfaces must have the same Kodaira dimension.

Proceeding, we also recall that a complex surface is said to be *minimal* if it is not the blow-up of some other complex surface. Any compact complex surface M^4 can be obtained from some minimal complex surface X, called *minimal model* for M^4, by blowing up X at finite number of points. In particular, from Kodaira's classification theory a complex surface is of general type if and only if it has a minimal model with $c_1^2 > 0$ and $c_1 \cdot [\omega] < 0$ for some Kähler class, where c_1 is the first Chern class. For details we refer to [3] and [30]. The compact complex-hyperbolic 4-manifold $\mathbb{C}\mathscr{H}^2/\Gamma$ is a classical example of minimal compact complex surface of general type. Usually, the complex-hyperbolic plane $\mathbb{C}\mathscr{H}^2$ can be seen as the unit ball in $\mathbb{C}^2$ endowed with the Bergmann metric. Furthermore, it is well-known that $c_1^2(X) = \big(2\chi(X) + 3\tau(X)\big)$. In 2011, LeBrun [31] showed the following result via Seiberg-Witten theory.

Theorem 1.1 (LeBrun, [31])**.** *Let $M^4 = X \sharp j\overline{\mathbb{CP}}^2$ be a compact Kähler surface, where X is the minimal model of M^4. If the Yamabe invariant of M^4 is negative, then*

$$Vol_{K,s}(M^4) \geq \frac{9}{4} Vol_s(M^4).$$

Moreover, equality holds if M^4 is a compact complex-hyperbolic 4-manifold $\mathbb{C}\mathscr{H}^2/\Gamma$.

It is important to be emphasized that $\mathrm{Vol}_s(M) = \frac{2\pi^2}{9} c_1^2(X)$, where X is the minimal model for M^4. For details, we refer to [27].

The primary purpose of the present article is to provide new estimates to Yamabe minimal volume and mixed minimal volume involving some topological invariants on four-dimensional compact manifolds. Moreover, we obtain topological sphere theorems for compact submanifolds of spheres and Euclidean spaces such that the full norm of the second fundamental form is bounded. Further, we obtain some topological obstructions to the existence of Einstein metrics on four-dimensional compact manifolds.

In order to explain our results to follow let us fix notation. First of all, for each plane $P \subset T_xM$ at a point $x \in M^4$, we recall that the *biorthogonal (sectional) curvature* of P is given by the following average of the sectional curvatures

$$K^{\perp}(P) = \frac{K(P) + K(P^{\perp})}{2}, \tag{1.6}$$

where $P^{\perp}$ is the orthogonal plane to P. In particular, for each point $x \in M^4$, we take the minimum of biorthogonal curvature to obtain the following function

$$K_1^{\perp}(x) = \min\{K^{\perp}(P); P \text{ is a 2- plane in } T_xM\}. \tag{1.7}$$

One should be emphasized that the sum of pair of sectional curvatures on two orthogonal planes, which was perhaps first observed by Chern [13], plays a crucial role in dimension four. Indeed, the positivity of the biorthogonal curvature is an intermediate condition between positive sectional curvature and positive scalar curvature. So, it is expected to get interesting results from this approach. For instance, Gray [16] showed that Euler characteristic $\chi(M)$ of a four-dimensional compact oriented manifold is nonnegative, provided that the sectional curvature satisfies

$\frac{K(P^{\perp})}{K(P)} \geq \frac{3}{4}$ whenever $K(P) \neq 0$. Moreover, as it was remarked by Singer and Thorpe [44], a four-dimensional Riemannian manifold (M^4, g) is Einstein if and only if $K^{\perp}(P) = K(P)$ for any plane $P \subset T_xM$ at any point $x \in M^4$. From Seaman [43] and Costa and Ribeiro [9], $\mathbb{S}^4$ and $\mathbb{CP}^2$ are the only compact simply-connected four-dimensional manifolds with positive biorthogonal curvature that can have (weakly) $1/4$-pinched biorthogonal curvature, or nonnegative isotropic curvature, or satisfy $K^{\perp} \geq \frac{s}{24} > 0$. While Bettiol [7] proved that the positivity of biorthogonal curvature is preserved under connected sums. In particular, he showed that $\mathbb{S}^4$, $\sharp^m\mathbb{CP}^2\sharp^n\overline{\mathbb{CP}}^2$ and $\sharp^n\left(\mathbb{S}^2 \times \mathbb{S}^2\right)$ admit metrics with positive biorthogonal curvature. For more details see [6, 7, 9, 36, 37] and [42].

At same time, it is important to recall some facts reached in [10, 11]. Based on ideas developed by Lebrun and Gursky [19] in the study of the modified Yamabe problem, Cheng and Zhu [12] studied the modified Yamabe problem in terms of a functional depending on Weyl curvature tensor (see also Section 2.2 in [35]), which is another important tool in the present work. While Itoh [22] described the global geometry of the modified scalar curvature. Moreover, Costa et al. [10] studied the minimal volume and minimal curvature on four-dimensional manifolds by means of the notion of biorthogonal curvature combined with the modified Yamabe problem. In particular, they showed that $12K_1^{\perp}$ is a modified scalar curvature as well as $Y_1^{\perp}(M^4,[g])$ given by

$$Y_1^{\perp}(M^4,[g]) = \inf_{\overline{g}\in[g]} \frac{12}{Vol(M,g)^{\frac{1}{2}}} \int_M \overline{K}_1^{\perp} dV_{\overline{g}}, \tag{1.8}$$

is a *modified Yamabe constant*, where $[g]$ denotes the conformal class of g. This implies that

$$Y_1^{\perp}(M) = \sup_{g\in\mathscr{M}} Y_1^{\perp}(M,[g]) \tag{1.9}$$

is the associated *modified Yamabe invariant*. It follows immediately that $Y_1^{\perp}(M) \leq Y(M)$, where $Y(M)$ denotes the standard Yamabe invariant of M^4. We further highlight that

(1) $Y_1^{\perp}(\mathbb{C}\mathscr{H}^2/\Gamma) \geq \frac{1}{6}Y(\mathbb{C}\mathscr{H}^2/\Gamma)$,
(2) $Y_1^{\perp}(\mathbb{H}^2 \times \mathbb{H}^2/\Gamma) \geq \frac{1}{4}Y(\mathbb{H}^2 \times \mathbb{H}^2/\Gamma)$ and
(3) $Y_1^{\perp}(\mathbb{H}^4/\Gamma) = Y(\mathbb{H}^4/\Gamma) = -8\pi\sqrt{3\chi(\mathbb{H}^4/\Gamma)}$.

Here, we use the notion of biorthogonal curvature (see also Eq. (2.7) in Section 2) to define a minimal volume invariant of M^4 by setting

$$\mathrm{Vol}_{K_1^{\perp},s}(M) = \inf\left\{Vol(M,g);\ \frac{1}{2}\left(K_1^{\perp} + \frac{s}{12}\right) \geq -1\right\}. \tag{1.10}$$

The key ingredient here that should be emphasized is that $\frac{1}{2}\left(K_1^{\perp} + \frac{s}{12}\right)$ is a modified scalar curvature; see [10] and [22]. This fact plays a fundamental role in the proof of our first result. In particular, we consider $Y_{K_1^{\perp},s}(M)$ to be its corresponding modified Yamabe invariant. With these notations in mind and motivated by

remarkable ideas drawn by Gursky [20] and LeBrun [31] we are in position to state our first result, which can be compared with Theorem 1.1.

Theorem 1.2. *Let $M^4 = X\sharp j\mathbb{CP}^2$ be a compact Kähler surface, where X is the minimal model of M^4. We assume that $Y(M) < 0$. Then we have:*

$$\mathrm{Vol}_{K,s}(M) \geq \mathrm{Vol}_{K_1^\perp,s}(M) \geq |Y_{K_1^\perp,s}(M)|^2 \geq \frac{9}{4}\mathrm{Vol}_s(M). \tag{1.11}$$

Moreover, we have equalities if M^4 is a compact complex-hyperbolic 4-manifold $\mathbb{C}\mathscr{H}^2/\Gamma$.

As it was previously mentioned a four-dimensional Riemannian manifold (M^4, g) is Einstein if and only if $K^\perp = K$. Therefore, we easily see that $\mathrm{Vol}_{K,s}(M) = \mathrm{Vol}_{K_1^\perp,s}(M) = \frac{9}{4}\mathrm{Vol}_s(M)$ for any compact complex-hyperbolic 4-manifold $\mathbb{C}\mathscr{H}^2/\Gamma$. Further, it is worth pointing out that

$$\mathrm{Vol}_{K_1^\perp,s}(\mathbb{H}^2\times\mathbb{H}^2/\Gamma) \leq \frac{8\pi^2}{3}\chi(\mathbb{H}^2\times\mathbb{H}^2/\Gamma). \tag{1.12}$$

Indeed, we consider $\mathbb{H}^2\times\mathbb{H}^2/\Gamma$ endowed with its canonical Kähler-Einstein metric. In this case, we have $K_1^\perp = \frac{s}{4}$ and $|W^+|^2 = |W^-|^2 = \frac{s^2}{24}$. Therefore, by using Chern-Gauss-Bonnet formula (2.4) we infer

$$96\pi^2\chi\big(\mathbb{H}^2\times\mathbb{H}^2/\Gamma\big) = s^2 Vol\big(\mathbb{H}^2\times\mathbb{H}^2/\Gamma\big).$$

Choosing the normalized scalar curvature $s=-6$ we obtain $\frac{1}{2}\Big(s+12K_1^\perp\Big) = -12$. For this, we arrive at (1.12).

In [32], LeBrun considered $\mathscr{C}\subset H^2(M,\mathbb{R})$ to be the set of *monopole classes*, which are the first Chern classes of those $spin^c$ structures on M^4 for which the Seiberg-Witten equation have solution for all metrics. In particular, its convex hull, denoted by $\mathbf{Hull}(\mathscr{C})$, is compact. From this, he introduced a real-valued invariant of M^4 by setting

$$\beta^2(M) = \max\left\{\int_M \alpha\wedge\alpha dV_g;\ \alpha\in\mathbf{Hull}(\mathscr{C})\right\} \tag{1.13}$$

if $\mathscr{C}$ is not empty. Otherwise, we set $\beta^2(M) = 0$. There are many four-dimensional manifolds M^4 for which $\beta^2(M) > 0$. With these definitions, LeBrun showed that a four-dimensional compact oriented manifold with $b_2^+ \geq 2$ satisfies

$$\int_M s^2 dV_g \geq 32\pi^2\beta^2(M). \tag{1.14}$$

In addition, if $\mathscr{C}$ is not empty, we have

$$Y(M) \leq -4\pi\sqrt{2\beta^2(M)}. \tag{1.15}$$

We refer to [15, 23, 32] for more details on monopole classes.

It is possible to show a relation between the LeBrun's invariant $\beta^2(M)$ and the modified scalar curvature $\frac{1}{2}\big(K_1^\perp + \frac{s}{12}\big)$ as well as its corresponding modified Yamabe invariant $Y_{K_1^\perp,s}(M)$. In this sense, we have the following result.

Theorem 1.3. *Let M^4 be a 4-dimensional compact oriented manifold. Suppose that $\mathscr{C}$ is not empty and $b_2^+(M) \geq 2$. Then, for any metric g on M^4, we have:*

$$\int_M \Big[\frac{1}{2}\big(K_1^{\perp} + \frac{s}{12}\big)\Big]^2 dV_g \geq \frac{\pi^2}{4}\beta^2(M). \tag{1.16}$$

In particular, the equality is attained by $\mathbb{C}\mathscr{H}^2/\Gamma$. In addition, if $Y(M) < 0$, then

$$|Y_{K_1^{\perp},s}(M)|^2 \geq 36\pi^2\beta^2(M). \tag{1.17}$$

1.2. **Sphere Theorem for Submanifolds.** Arguably, it is very interesting to investigate curvature and topology of submanifolds of spheres and Euclidean spaces. In this sense, in 1973, Lawson and Simons [26], by means of nonexistence for stable currents on compact submanifolds of a sphere, obtained a criterion for the vanishing of the homology groups of compact submanifolds of spheres. In particular, they showed the following result.

Theorem 1.4 (Lawson-Simons, [26])**.** *Let M^n be an $n(\geq 4)$-dimensional oriented compact submanifold in the unit sphere $\mathbb{S}^{n+p}$.*

(1) *If $n = 4$, and the second fundamental form α of M^4 satisfies $\|\alpha\|^2 < 3$, then M^4 is a homotopy sphere.*
(2) *If $n \geq 5$, and the second fundamental form α of M^n satisfies $\|\alpha\|^2 < 2\sqrt{n-1}$, then M^n is homeomorphic to a sphere.*

Afterward, Leung [33] and Xin [50] were able to extend the results obtained by Lawson and Simons to compact submanifolds of Euclidean spaces. In 2001, inspired by ideas developed in [26, 33] and [50], Asperti and Costa [2] obtained an estimate for the Ricci curvature of submanifolds of a space form which improves Leung's estimates. As a consequence, Asperti and Costa obtained a new criterion for the vanishing of the homology groups of compact submanifolds of spheres and Euclidean spaces. In 2009, Xu and Zhau [51] investigated the topological and differentiable structures of submanifolds by imposing certain conditions on the second fundamental form. While Gu and Xu [18] used the convergence results obtained by Hamilton and Brendle as well as Lawson-Simons-Xin formulae to obtain a differentiable sphere theorem for submanifolds in space forms. Similar result was obtained by Andrews-Baker [1] making use of the mean curvature flow. For a comprehensive references on such a subject, we indicate, for instance [2, 18, 26, 33] and [51].

In order to announce our next results let us set up notation. Firstly, we denote by $f : M^n \to \mathscr{Q}_c^{n+m}$ an isometric immersion of a connected n-dimensional compact Riemannian manifold M^n into a complete, simply connected $(n+m)$-dimensional manifold $\mathscr{Q}_c^{n+m}$ with constant sectional curvature c. We denote by T_pM the tangent space of M^n, for each point $p \in M^n$, and $\big(T_pM\big)^{\perp}$ stands for the normal space of the immersion f at point p. Also, $\vec{H}$ and $\alpha : T_pM \times T_pM \to \big(T_pM\big)^{\perp}$ stand for the mean curvature vector and the second fundamental form of the immersion, respectively. We adopt the following convention: If $p \in M^n$ is such that $\vec{H}(p) \neq 0$, then $\lambda_1 \leq \ldots \leq \lambda_n$ denote the eigenvalues of the Weingarten operator A_{ξ_1} with $\xi_1 = \frac{1}{H}\vec{H}(p)$,

where $H = \|\vec{H}\|$ is the length of the mean curvature vector. Otherwise, if $\vec{H}(p) = 0$, we then pick $\lambda_i = 0$ for $1 \leq i \leq n$, and ξ_1 any unit vector normal to M^n at point p. The following result, due to Asperti and Costa [2], is very important for our purposes.

Theorem 1.5 (Asperti-Costa, [2])**.** *Let $f : M^n \to \mathcal{Q}_c^{n+m}$ be an isometric immersion, where M^n is a compact oriented manifold and $c \geq 0$. If for some integer p satisfying $2 \leq p \leq \frac{n}{2}$*

$$\|\alpha\|^2 < \frac{n^2H^2}{(n-p)} + \frac{n(n-2p)H\lambda_1}{(n-p)} + nc \tag{1.18}$$

holds on M^n, then the k-th homology group $H_k(M,\mathbb{Z}) = 0$, for $p \leq k \leq n-p$.

Here, motived by [1, 2] and [26], we shall obtain topological sphere theorems for compact submanifolds of spheres and Euclidean spaces, provided that the full norm of the second fundamental form is bounded by a fixed multiple of the length of the mean curvature vector. More precisely, we may announce our next result as follows.

Theorem 1.6. *Let M^4 be a connected 4-dimensional oriented compact submanifold of $\mathcal{Q}_c^{4+m}$ with $c \geq 0$. Then we have:*

$$4K_1^{\perp} \geq -\|\alpha\|^2 + 4\big(2H^2 + c\big). \tag{1.19}$$

In particular, if M^4 has finite fundamental group and $\|\alpha\|^2 < 4\big(2H^2+c\big)$, then M^4 is homeomorphic to a sphere $\mathbb{S}^4$.

It is important to highlight that our estimate obtained in Theorem 1.6 improves the estimate obtained in Theorem 4 in [18]. Moreover, it can be seen as a generalization, in the topological sense, of Theorem 4 [18]. We also point out that the condition $\|\alpha\|^2 < \frac{n^2H^2}{n-1} + 2c$ used in Theorem 4 in [18] implies that M^4 has nonnegative sectional curvature.

In the sequel, by means of Theorem 1.5 we obtain the following result.

Theorem 1.7. *Let M^4 be a 4-dimensional oriented compact submanifold in the unit sphere $\mathbb{S}^{4+m}$, $m \geq 1$.*

1. *If $\|\alpha\|^2 < 4$, then M^4 has positive biorthogonal curvature. In addition, if M^4 has finite fundamental group, then M^4 is homeomorphic to a sphere $\mathbb{S}^4$.*
2. *If $\|\alpha\|^2 \leq 4$, then M^4 has nonnegative biorthogonal curvature. Moreover, if for every point of M^4 some biorthogonal curvature vanishes, then M^4 is a minimal submanifold of $\mathbb{S}^n$. In addition, if $m = 1$, then M^4 must be $\mathbb{S}^2_{c_1} \times \mathbb{S}^2_{c_2}$.*

We highlight that Theorem 1.7 can be seen as an improvement of Theorem 1.4 in dimension four. Also, we notice that the results obtained in second item of Theorem 1.7 generalizes [34].

1.3. **Topological Obstruction.** In the remainder part of this section we investigate topological obstructions on four-dimensional manifolds. In the sequel we recall that (M^4, g) is called *Einstein* if the Ricci curvature is given by

$$Ric = \lambda g,$$

for some constant λ. A result due to Berger [4] combined with Synge's theorem tells us that, if M^4 is an Einstein manifold with positive sectional curvature, then it satisfies $2 \leq \chi(M) \leq 9$. In classical articles Hitchin [21] and Thorpe [47] have proved, independently, that if M^4 is an Einstein manifold, then

$$\chi(M) \geq \frac{3}{2}|\tau(M)|. \tag{1.20}$$

Moreover, Hitchin proved that the equality holds in (1.20) if and only if M^4 is diffeomorphic to $K3$ surfaces. The Hitchin-Thorpe inequality tells us that every Einstein metric on $K3$ is one of the hyper-Kähler metrics constructed by S.-T. Yau [52]; for more details see [5]. This inequality was, for a long time, the only known obstruction to the existence of Einstein metrics on four-dimensional manifolds. Interesting enough, motivated by Gromov's arguments, Kotschick [24] obtained an improvement of the Hitchin-Thorpe inequality. More precisely, he was able to show that a four-dimensional compact oriented Einstein manifolds satisfies

$$\chi(M) \geq \frac{3}{2}|\tau(M)| + \frac{1}{162\pi^2}\|M\|. \tag{1.21}$$

The idea behind the inequality is that in four dimensions the topological invariants $|\tau(M)|$ and $\chi(M)$ are given in terms of curvature, and therefore a Riemannian manifold with special curvature, such as an Einstein manifold, ought to satisfy some topological obstructions. We can therefore state our next result as follows.

Theorem 1.8. *Let M^4 be a 4-dimensional oriented compact manifold admitting an Einstein metric g. Suppose that $Y_1^{\perp}(M) \leq 0$. Then the following assertions hold:*

(1)
$$\chi(M) \geq \frac{1}{576\pi^2}|Y_1^{\perp}(M)|^2.$$
Moreover, if the equality holds, then M^4 is either flat or $\mathbb{H}^2_c \times \mathbb{H}^2_c/\Gamma$.

(2) *If g is half-conformally flat, then*
$$\chi(M) \geq \frac{1}{384\pi^2}|Y_1^{\perp}(M)|^2.$$
In particular, if the equality holds, then M^4 is a compact complex-hyperbolic 4-manifold $\mathbb{C}\mathscr{H}^2/\Gamma$.

Taubes [48] proved that for any smooth compact oriented four-dimensional manifold X, there is a integer j such that $M^4 = X\sharp j\overline{\mathbb{CP}}^2$ admits half-conformally flat metrics. The minimal number for j is called *Taubes Invariant*, which is unknown for most four-dimensional manifolds. At same time, we have the following result.

Corollary 1. $\mathbb{H}^2_c \times \mathbb{H}^2_c/\Gamma\sharp j(\mathbb{S}^1 \times \mathbb{S}^3)$ *does not admit Einstein metric provided that* $j > \frac{4}{9}\chi(\mathbb{H}^2_c \times \mathbb{H}^2_c/\Gamma)$.

2. Preliminaries and Notations

In this section we shall present some preliminaries and notations that will be useful for the establishment of the desired results.

2.1. **Four-Dimensional Manifolds.**

As it was previously remarked four-dimensional manifolds are fairly special. Indeed, many peculiar features are directly attributable the fact that the bundle of 2-forms on a four-dimensional oriented Riemannian manifold can be invariantly decomposed as a direct sum

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-, \tag{2.1}$$

where $\Lambda^\pm$ is the (± 1)-eigenspace of Hodge star operator $*$. The decomposition (2.1) is conformally invariant. Moreover, it allows us to conclude that the Weyl tensor W is an endomorphism of $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ such that $W = W^+ \oplus W^-$. A manifold is *locally conformally flat* if $W = 0$. It is said *half-conformally flat* if either $W^- = 0$ or $W^+ = 0$. Furthermore, an oriented manifold is *self-dual* if $W^- = 0$. On a half-conformally flat manifold, self-duality is a property that depends on the orientation. The complex projective space $\mathbb{CP}^2$ endowed with Fubini-Study metric, for instance, shows that, in real dimension 4, the half-conformally flat condition is really weaker than locally conformally flat condition. Since the Riemann curvature tensor $\mathscr{R}$ of M^4 can be seen as a linear map on Λ^2, we have the following decomposition

$$\mathscr{R} = \left(\begin{array}{c|c} W^+ + \frac{s}{12} Id & \mathring{Ric} \\ \hline \mathring{Ric}^* & W^- + \frac{s}{12} Id \end{array} \right), \tag{2.2}$$

where $\mathring{Ric}$ is the Ricci traceless.

Next, let $H^\pm(M^4, \mathbb{R})$ be space of positive and negative harmonic 2-forms, respectively. Therefore, the second Betti number b_2 of M^4 can be written as $b_2 = b_2^+ + b_2^-$, where $b_2^\pm$ = dim $H^\pm(M^4, \mathbb{R})$. In particular, the signature of M^4 is given by

$$\tau(M) = b_2^+ - b_2^-.$$

Moreover, by the Hirzebruch signature theorem, it can be expressed as follows

$$12\pi^2 \tau(M) = \int_M \Big(|W^+|^2 - |W^-|^2 \Big) dV_g. \tag{2.3}$$

In addition, by Chern-Gauss-Bonnet formula, the Euler characteristic of M^4 can be written as

$$8\pi^2 \chi(M) = \int_M \Big(|W^+|^2 + |W^-|^2 + \frac{s^2}{24} - \frac{1}{2} |\mathring{Ric}|^2 \Big) dV_g. \tag{2.4}$$

We now fix a point and diagonalize $W^{\pm}$ such that $w_i^{\pm}$, $1 \leq i \leq 3$, are their respective eigenvalues. So, we remark that the eigenvalues of $W^{\pm}$ satisfy

$$w_1^{\pm} \leq w_2^{\pm} \leq w_3^{\pm} \text{ and } w_1^{\pm} + w_2^{\pm} + w_3^{\pm} = 0. \tag{2.5}$$

In particular, (2.5) tells us that

$$|W^{\pm}|^2 \leq 6(w_1^{\pm})^2. \tag{2.6}$$

Next, as it was explained in [9] and [42], Eq. (1.7) provides us the following useful identity

$$K_1^{\perp} = \frac{w_1^+ + w_1^-}{2} + \frac{s}{12}. \tag{2.7}$$

More details can be seen in [9].

Lemma 1. *Let $\alpha^+ \in H^+(M^4, \mathbb{R})$ be a non-degenerate harmonic 2-form on a 4-dimensional compact, oriented manifold M^4. Then we have:*

$$\int_M |\nabla \alpha^+|^2 dV_g \geq \frac{2}{3} \int_M \big(6K_1^{\perp} - s\big)|\alpha^+|^2 dV_g.$$

In particular, the equality holds if and only if $W^- = 0$ and α^+ belongs to the smallest eigenspace of W^+.

Proof. Since $\alpha^+ \in H^+(M, \mathbb{R})$ is a non-degenerate harmonic 2-form, we have the following Weitzenböch formulae

$$0 = \langle \Delta \alpha^+, \alpha^+ \rangle = \frac{1}{2}\Delta \mid \alpha^+ \mid^2 + \mid \nabla \alpha^+ \mid^2 + \langle (\frac{s}{3} - 2W^+)\alpha^+, \alpha^+ \rangle. \tag{2.8}$$

By using that w_1^+ is the smallest eigenvalue of W^+ we infer

$$\langle W^+(\alpha^+), \alpha^+ \rangle \geq w_1^+ \langle \alpha^+, \alpha^+ \rangle.$$

Upon integrating of (2.8) over M^4 we use the above information to deduce

$$\int_M |\nabla \alpha^+|^2 dV_g + \int_M \frac{s}{3}|\alpha^+|^2 dV_g \geq 2\int_M \big(w_1^+ + w_1^-\big)|\alpha^+|^2 dV_g. \tag{2.9}$$

By means of (2.7), it is easy to check that

$$\int_M |\nabla \alpha^+|^2 dV_g \geq \frac{2}{3} \int_M \big(6K_1^{\perp} - s\big)|\alpha^+|^2 dV_g,$$

as desired. □

In the sequel we recall that $w_1^+ : M^4 \to (-\infty, 0]$ is a Lipschitz continuos function. Then, following [31] we define $f_-(x) = min\{f(x), 0\}$, where f is a real-valued function on M^4. With these notations in mind we have the following proposition, which is a small modification of Proposition 2.2 in quoted article.

Proposition 1. *Let $M^4 = X \sharp j\mathbb{CP}^2$ be a compact Kähler surface, where X is the minimal model of M^4. We assume that $Y(M) < 0$. Then we have:*

$$\frac{\pi^2}{2}\big(2\chi(M) + 3\tau(M)\big) \leq \Big(\int_M |\frac{1}{2}(K_1^{\perp} + \frac{s}{12})_-|^3 dV_g\Big)^{\frac{2}{3}} Vol(M,g)^{\frac{1}{3}}. \tag{2.10}$$

Moreover, if (2.10) is actually an equality, then M^4 is half-conformally flat.

Proof. First, we apply the same arguments used by LeBrun [31] in the proof of Proposition 2.2 to arrive at

$$-\int_M \Big(\frac{2}{3}s+2w_1^+\Big)_-|\phi|^4 dV_g \geq \int_M (-s)|\phi|^2 dV_g - 4\int_M |\phi|^2|\nabla\phi|^2 dV_g, \tag{2.11}$$

where ϕ is a solution of the Seiberg-Witten equations. Moreover, we already know that

$$\int_M (-s)|\phi|^4 dV_g \geq 4\int_M |\phi|^2|\nabla\phi|^2 dV_g + \int_M |\phi|^6 dV_g, \tag{2.12}$$

for details see Eq. 7 in [31] (see also pg. 399 in [41]).

It follows immediately from (2.5) that

$$\frac{2}{3}s+2w_1^+ + 2w_1^- \leq \frac{2}{3}s+2w_1^+, \tag{2.13}$$

with equality if and only if M^4 is half-conformally flat. Then, we combine (2.11), (2.12), (2.7) and (2.13) to get

$$-\int_M \big(\frac{s}{3}+4K_1^\perp\big)_-|\phi|^4 dV_g \geq \int_M |\phi|^6 dV_g. \tag{2.14}$$

On the other hand, we use the Hölder inequality to infer

$$-\int_M \big(\frac{s}{3}+4K_1^\perp\big)_-|\phi|^4 dV_g \leq \Big(\int_M \big|\big(\frac{s}{3}+4K_1^\perp\big)_-\big|^3 dV_g\Big)^{\frac{1}{3}}\Big(\int_M |\phi|^6 dV_g\Big)^{\frac{2}{3}} \tag{2.15}$$

and

$$\int_M |\phi|^6 dV_g \geq Vol(M,g)^{-\frac{1}{2}}\Big(\int_M |\phi|^4 dV_g\Big)^{\frac{3}{2}}. \tag{2.16}$$

Combining Eqs. (2.14) and (2.15) we deduce

$$\int_M |\phi|^6 dV_g \leq \int_M \big|\big(\frac{s}{3}+4K_1^\perp\big)_-\big|^3 dV_g. \tag{2.17}$$

Furthermore, we use (2.16) to infer

$$\int_M |\phi|^4 dV_g \leq \Big(\int_M \big|\big(\frac{s}{3}+4K_1^\perp\big)_-\big|^3 dV_g\Big)^{\frac{2}{3}} Vol(M,g)^{\frac{1}{3}}. \tag{2.18}$$

On the other hand, we already know from Seiberg-Witten theory (cf. Eq. 4 in [31]) that

$$32\pi^2\big(2\chi(M)+3\tau(M)\big) \leq \int_M |\phi|^4 dV_g. \tag{2.19}$$

We then combine (2.18) with (2.19) to arrive at (2.10). Moreover, note that if (2.10) is equality we have $w_1^- \equiv 0$, and this case we can use (2.5) to conclude that M^4 must be half-conformally flat, which establishes the proposition. $\square$

2.2. **Additional Notations.** From now on, we set up some notations that will be useful in the proofs of Theorems 1.6 and 1.7. Firstly, we consider M^n to be an n-dimensional compact submanifold in an $(n+m)$-dimensional Riemannian manifold N^{n+m}. We adopt the following convention on the indices:

$$1 \leq i,\, j, k \leq n \text{ and } n+1 \leq \beta,\, \gamma \leq n+m.$$

We then denote by R_{ijkl}, $\overline{R}_{ijkl}$, α and $\vec{H}$ stand for the Riemannian curvature tensor of M^4, Riemannian curvature tensor of N^{n+m}, second fundamental form and mean curvature vector, respectively. Therefore, we have:

$$R_{ijkl} = \overline{R}_{ijkl} + \sum_{\beta}\left(\alpha^{\beta}_{ik}\alpha^{\beta}_{jl} - \alpha^{\beta}_{il}\alpha^{\beta}_{jk}\right)$$

and

$$R_{\beta\gamma kl} = \overline{R}_{\beta\gamma kl} + \sum_{i}\left(\alpha^{\beta}_{ik}\alpha^{\gamma}_{jl} - \alpha^{\beta}_{il}\alpha^{\gamma}_{jk}\right).$$

We further have

$$\vec{H} = \frac{1}{n}\sum_{\beta, i}\alpha^{\beta}_{ii}e_{\beta}.$$

From Gauss Equation we infer

$$Ric(e_i) = \sum_{j}\overline{R}_{ijij} + \sum_{\beta, j}\left[\alpha^{\beta}_{ii}\alpha^{\beta}_{jj} - \left(\alpha^{\beta}_{ij}\right)^2\right]. \tag{2.20}$$

Moreover, if N has constant sectional curvature c, then the scalar curvature of M^n is given by

$$s = n(n-1)c + n^2H^2 - \|\alpha\|^2, \tag{2.21}$$

where H is the length of the mean curvature vector.

3. Proof of the Main Results

3.1. **Proof of Theorem 1.2.**

Proof. To begin with, for any function ϕ on M^4 we set $\overline{g} = e^{2\phi}g \in [g]$. From this, it follows that

$$\overline{s} = e^{-2\phi}(-6\Delta\phi - 6|\nabla\phi|^2 + s)$$

and

$$6e^{2\phi}\overline{K}^{\perp}_1 = 6K^{\perp}_1 - 3\Delta\phi - 3|\nabla\phi|^2,$$

which reduces to

$$e^{2\phi}\left(\overline{K}^{\perp}_1 + \frac{\overline{s}}{12}\right) = K^{\perp}_1 - \Delta\phi - |\nabla\phi|^2 + \frac{s}{12}.$$

On the other hand, note that $s + 3(w^+_1 + w^-_1) = \frac{1}{2}\left(K^{\perp}_1 + \frac{s}{12}\right)$ is a modified scalar curvature (cf. [10] and [22]). Hence, since $Y_{s,K^{\perp}_1}(M) \leq Y(M) < 0$, we can invoke Itoh's theorem [22] to deduce that there is a metric $g \in [g]$ for which the modified

scalar curvature $\frac{1}{2}\Big(K_1^{\perp}+\frac{s}{12}\Big)$ is a constant negative. Therefore, a straightforward computation using these informations and Stokes formula yields

$$\int_M |K_1^{\perp}+\frac{s}{12}|dV_g \leq \int_M e^{2\phi}|\overline{K}_1^{\perp}+\frac{\overline{s}}{12}|dV_g. \tag{3.1}$$

We then use the Cauchy-Schwarz inequality as well as $dV_{\overline{g}}=e^{4\phi}dV_g$ to get

$$\frac{1}{Vol(M,g)}\Big(\int_M |\frac{1}{2}\Big(K_1^{\perp}+\frac{s}{12}\Big)|dV_g\Big)^2 \leq \int_M |\frac{1}{2}\Big(\overline{K}_1^{\perp}+\frac{\overline{s}}{12}\Big)|^2 dV_{\overline{g}}, \tag{3.2}$$

Now, we use Proposition 1 to infer

$$\begin{aligned}\int_M |\frac{1}{2}\Big(\overline{K}_1^{\perp}+\frac{\overline{s}}{12}\Big)|^2 dV_{\overline{g}} &\geq [Vol(M,g)]^{\frac{1}{3}}\Big(\int_M |\frac{1}{2}\Big(K_1^{\perp}+\frac{s}{12}\Big)|^3 dV_g\Big)^{\frac{2}{3}}\\ &\geq 2\pi^2\big(2\chi(M)+3\tau(M)\big).\end{aligned} \tag{3.3}$$

Since $Y(M)<0$, we immediately deduce $\frac{1}{2}\Big(\overline{K}_1^{\perp}+\frac{\overline{s}}{12}\Big)\geq -1$. Now, without loss of generality, we may consider g instead $\overline{g}$. So, we notice that the

$$\inf_{g\in\mathscr{M}_{K_1^{\perp},s}}\int_M |\frac{1}{2}\Big(K_1^{\perp}+\frac{s}{12}\Big)|^2 dV_g \geq \inf_{g\in\mathscr{M}}\int_M |\frac{1}{2}\Big(K_1^{\perp}+\frac{s}{12}\Big)|^2 dV_g,$$

where $\mathscr{M}_{K_1^{\perp},s}$ is the set of metrics such that $\big(K_1^{\perp}+\frac{s}{12}\big)\geq -1$. From here it follows that

$$\mathrm{Vol}_{K,s}(M^4)\geq \mathrm{Vol}_{K_1^{\perp},s}(M^4)\geq \inf_{g\in\mathscr{M}}\int_M |\frac{1}{2}\Big(K_1^{\perp}+\frac{s}{12}\Big)|^2 dV_g \geq \frac{9}{4}\mathrm{Vol}_s(M^4).$$

From now on it is suffices to repeat the final arguments of the proof of Lemma 2.6 in [22] (see also Proposition 3 in [10]) to deduce

$$|Y_{K_1^{\perp},s}(M)|^2 = \inf_{g\in\mathscr{M}}\int_M |\frac{1}{2}\Big(K_1^{\perp}+\frac{s}{12}\Big)|^2 dV_g,$$

which finishes the proof of the theorem. □

3.2. **Proof of Theorem 1.3.**

Proof. The proof will follow the trend developed by LeBrun in [32]. Initially, we invoke Lemma 3.7 of [32] to deduce that, for any smooth positive function f on M^4, the rescaled Seiberg-Witten equation have solution (ϕ, A). Next, we remember that

$$0\geq \int_M \Big[4|\phi|^2|\nabla_A\phi|^2+s|\phi|^4+f|\phi|^6\Big]dV_g$$

(cf. Eq. (19) in [32]). Now, setting $\psi = 2\sqrt{2}\sigma(\phi)$ and invoking Lemma 1 we arrive at

$$0\geq \int_M \Big[s|\psi|^2+f|\psi|^3\Big]dV_g+\frac{2}{3}\int_M\Big(6K_1^{\perp}-s\Big)|\psi|^2 dV_g,$$

which can be written succinctly as

$$0 \geq 4\int_M \left[K_1^{\perp} + \frac{s}{12}\right]|\psi|^2 dV_g + \int_M f|\psi|^3 dV_g.$$

We then set $\gamma = \frac{1}{4}\psi$ to infer

$$-\int_M \left(K_1^{\perp} + \frac{s}{12}\right)|\gamma|^2 dV_g \geq \int_M f|\gamma|^3 dV_g,$$

which can rewritten as

$$\int_M \left[-\left(K_1^{\perp} + \frac{s}{12}\right) f^{-\frac{2}{3}}\right]\left[f^{\frac{2}{3}}|\gamma|^2\right] dV_g \geq \int_M f|\gamma|^3 dV_g.$$

By the Hölder inequality, it now follows that

$$\int_M |K_1^{\perp} + \frac{s}{12}|^3 f^{-2} dV_g \geq \int_M f|\gamma|^3 dV_g \tag{3.4}$$

On the other hand, the Hölder inequality ensures

$$\left(\int_M f^4 dV_g\right)^{\frac{1}{3}}\left(\int_M f|\gamma|^3 dV_g\right)^{\frac{2}{3}} \geq \int_M f^{\frac{4}{3}}\left(f^{\frac{2}{3}}|\gamma|^2\right) dV_g. \tag{3.5}$$

This jointly with (3.4) yields

$$\left(\int_M f^4 dV_g\right)^{\frac{1}{3}}\left(\int_M |K_1^{\perp} + \frac{s}{12}|^3 f^{-2}\right)^{\frac{2}{3}} \geq \int_M f^2|\gamma|^2 dV_g. \tag{3.6}$$

Note that $\gamma = \frac{\sqrt{2}}{2}\sigma(\phi)$ and thus $f\gamma = \frac{\sqrt{2}}{2}(-iF_A^+)$. Then, from Lemma 1 and Proposition 4.5 in [32] we get

$$\left(\int_M f^4 dV_g\right)^{\frac{1}{2}}\left(\int_M |K_1^{\perp} + \frac{s}{12}|^3 f^{-2}\right)^{\frac{2}{3}} \geq 2\pi^2\beta^2(M),$$

for any smooth function f on M^4.

From now on choose a decreasing sequence of smooth positive functions f_k on M^4 such that

$$\lim_{k\to\infty} f_k^2 = \frac{1}{2}\left|K_1^{\perp} + \frac{s}{12}\right|$$

uniformly on M^4. With this setting we have

$$\int_M \left[\frac{1}{2}\left(K_1^{\perp} + \frac{s}{12}\right)\right]^2 dV_g \geq \frac{\pi^2}{4}\beta^2(M),$$

as asserted.

Next, since $Y(M) < 0$ we can invoke once more Proposition 3 in [10] to conclude the proof of the theorem. $\square$

3.3. **Proof of Theorem 1.6.**

Proof. Initially, for each point $p \in M^4$, we consider $\{v_1, v_2, v_3, v_4\}$ an orthonormal basis of T_pM, and such that $\{\xi_1, \xi_2, ..., \xi_m\}$ is an orthonormal referential in $\left(T_pM\right)^{\perp}$. So, the Weingarten operator A_{ξ_β}, in the normal direction ξ_β, is given by

$$\langle A_{\xi_\beta} v_i, v_j\rangle = \langle \alpha(v_i, v_j), \xi_\beta\rangle,$$

where $v_i, v_j \in T_pM$. From this, we have

$$\vec{H} = \frac{1}{4}\sum_{\beta \geq 1}\left(trA_{\xi_\beta}\right)\xi_\beta$$

and $H = \|\vec{H}\|$. Moreover, we have

$$\|\alpha\|^2 = \sum_{\beta\geq 1} trA^2_{\xi_\beta}.$$

We set $A_1 = A_{\xi_1}$ to be the Weingarten operator of f in the normal direction $\xi_1 = \frac{1}{H}\vec{H} \in (T_pM)^{\perp}$. We then notice that $trA_1 = 4H$ and $trA_\beta = 0$ for $\beta \geq 2$. In particular, if $X, Y \in \left(T_pM\right)^{\perp}$, then we get

$$\alpha(X,Y) = \sum_{\beta\geq 1}\langle A_\beta X, Y\rangle \xi_\beta.$$

It follows from Gauss Equation that

$$Ric(v_1) = \sum_{\beta\geq 1}\langle A_\beta v_1, v_1\rangle \sum_{i\neq 1}\langle A_\beta v_i, v_i\rangle - \sum_{\beta\geq 1}\sum_{i\neq 1}\langle A_\beta v_i, v_1\rangle^2 + 3c.$$

In analogous way we obtain

$$Ric(v_2) = \sum_{\beta\geq 1}\langle A_\beta v_2, v_2\rangle \sum_{i\neq 2}\langle A_\beta v_i, v_i\rangle - \sum_{\beta\geq 1}\sum_{i\neq 2}\langle A_\beta v_i, v_2\rangle^2 + 3c.$$

A straightforward computation by combining these two above equations arrives

$$\begin{aligned} Ric(v_1) + Ric(v_2) &= 4H\Big[\langle A_1 v_1, v_1\rangle + \langle A_1 v_2, v_2\rangle\Big] \\ &\quad - \sum_{\beta\geq 1}\Big[\langle A_\beta v_1, v_1\rangle^2 + \langle A_\beta v_2, v_2\rangle^2\Big] \\ &\quad - \sum_{\beta\geq 1}\Big[\sum_{i\neq 1}\langle A_\beta v_i, v_1\rangle^2 + \sum_{i\neq 2}\langle A_\beta v_i, v_2\rangle^2\Big] + 6c. \end{aligned} \tag{3.7}$$

Next, it is easy to see that

$$\begin{aligned} Ric(v_1) + Ric(v_2) &= -\Big[\langle A_1 v_1, v_1\rangle + \langle A_1 v_2, v_2\rangle - 2H\Big]^2 \\ &\quad - \sum_{\beta\geq 1}\Big[\sum_{i\neq 1}\langle A_\beta v_i, v_1\rangle^2 + \sum_{i\neq 2}\langle A_\beta v_i, v_2\rangle^2\Big] + 4H^2 + 6c \\ &\quad + 2\sum_{\beta\geq 1}\langle A_\beta v_1, v_1\rangle\langle A_\beta v_2, v_2\rangle - \sum_{\beta\geq 2}\Big[\langle A_\beta v_1, v_1\rangle + \langle A_\beta v_2, v_2\rangle\Big]^2. \end{aligned} \tag{3.8}$$

We then invoke once more Gauss Equation to infer

$$K(v_1,v_2)=\sum_{\beta\geq 1}\langle A_\beta v_1,v_1\rangle\langle A_\beta v_2,v_2\rangle-\sum_{\beta\geq 1}\langle A_\beta v_1,v_2\rangle^2+c.$$

This tell us that

$$Ric(v_1)+Ric(v_2)\leq 2K(v_1,v_2)+4c+4H^2+a+2\sum_{\beta\geq 1}\langle A_\beta v_1,v_2\rangle^2,$$

where $a=\sum_{\beta\geq 1}\Big[\sum_{i\neq 1}\langle A_\beta v_i,v_1\rangle^2+\sum_{i\neq 2}\langle A_\beta v_i,v_2\rangle^2\Big]$. In particular, we have

$$\begin{aligned}-\sum_{\beta\geq 1}\Big[\sum_{i\neq 1}\langle A_\beta v_i,v_1\rangle^2 &+ \sum_{i\neq 2}\langle A_\beta v_i,v_2\rangle^2\Big]+2\sum_{\beta\geq 1}\langle A_\beta v_1,v_2\rangle^2\\ &= -\sum_{\beta\geq 1,i\neq 1,2}\Big[\langle A_\beta v_i,v_1\rangle^2+\langle A_\beta v_i,v_2\rangle^2\Big]\leq 0.\end{aligned}$$

From here it follows that

$$2K(v_1,v_2)\geq Ric(v_1)+Ric(v_2)-4H^2-4c.$$

In a similar way we get

$$2K(v_3,v_4)\geq Ric(v_3)+Ric(v_4)-4H^2-4c.$$

This immediately gives

$$4K^{\perp}(P)\geq s-8H^2-8c,$$

where s denotes the scalar curvature of M^4 and P is the 2-plane generated by v_1,v_2. Therefore, we have $4K_1^{\perp}\geq s-8H^2-8c$. This together with (2.21) gives

$$4K_1^{\perp}\geq-\|\alpha\|^2+8H^2+4c,$$

as claimed.

Proceeding, we assume that $\|\alpha\|^2<4\big(2H^2+c\big)$. In this case, we invoke Theorem 1.5 to conclude $H_2(M,\mathbb{Z})=0$. Therefore, it suffices to apply Lemma 2.2 in [2] to conclude that M^4 is homeomorphic to sphere $\mathbb{S}^4$. This what we wanted to prove. $\square$

3.4. **Proof of Theorem 1.7.**

Proof. In order to prove the first assertion we assume that M^4 is isometrically immersed into $\mathbb{S}^{4+m}$. We already know from Theorem 1.6 that $4K_1^{\perp}\geq-\|\alpha\|^2+8H^2+$ 4 Moreover, since $\|\alpha\|^2<4$ we have $4K_1^{\perp}>8H^2\geq 0$. From here it follows that M^4 has positive biorthogonal curvature. In particular, it is easy to see that the estimate (1.18) holds (for $p=2$). Therefore, we may use Theorem 1.5 to deduce that $H_2(M,\mathbb{Z})=0$ and since M^4 has finite fundamental group, we can apply Lemma 2.2 in [2] to conclude that M^4 is homeomorphic to sphere $\mathbb{S}^4$, as desired.

Proceeding, since $\|\alpha\|^2\leq 4$ we immediately deduce from Theorem 1.6 that M^4 has nonnegative biorthogonal curvature. In particular, if for every point of M^4 some biorthogonal curvature vanishes, then $H=0$ and $\|\alpha\|^2=4$. Finally, if $m=1$ we invoke Chern-Do Carmo-Kobayashi theorem [14] to conclude that M^4 must be $\mathbb{S}^2_{c_1}\times\mathbb{S}^2_{c_2}$, which finishes the proof of the theorem.

□

3.5. **Proof of Theorem 1.8.**

Proof. We now begin by recalling that

$$|w_1^{\pm}|^2 \leq \frac{2}{3}|W^{\pm}|^2. \tag{3.9}$$

Moreover, the equality holds in (3.9) if and only if $w_3^{\pm} = w_2^{\pm}$ (cf. Lemma 3.2 (a) in [37]). This tells us that

$$|W^+|^2 + |W^-|^2 \geq \frac{3}{2}\left[(w_1^+)^2 + (w_1^-)^2\right] = \frac{1}{24}\left[(6w_1^+)^2 + (6w_1^-)^2\right].$$

Proceeding, since M^4 admits an Einstein metric g we may use Chern-Gauss-Bonnet formula (2.4) to arrive at

$$8\pi^2\chi(M) = \int_M \left(|W^+|^2 + |W^-|^2 + \frac{s^2}{24}\right) dV_g.$$

Whence, it follows immediately that

$$8\pi^2\chi(M) \geq \frac{1}{24}\int_M \left(s^2 + (6w_1^+)^2 + (6w_1^-)^2\right) dV_g.$$

Now, by the Cauchy-Schwarz inequality and (2.7) we infer

$$\begin{aligned} 8\pi^2\chi(M) &\geq \frac{1}{72}\int_M (s + 6w_1^+ + 6w_1^-)^2 dV_g \\ &= \frac{1}{72}\int_M 144(K_1^{\perp})^2 dV_g, \end{aligned}$$

which reduces to

$$4\pi^2\chi(M) \geq \int_M |K_1^{\perp}|^2 dV_g. \tag{3.10}$$

Moreover, if the equality holds, then $w_1^+ = w_1^- = \frac{s}{6}$ and $w_2^{\pm} = w_3^{\pm}$. In this case M^4 must be $\mathbb{H}_c^2 \times \mathbb{H}_c^2/\Gamma$.

Proceeding, taking into account that

$$4\pi^2\chi(M) \geq \inf_{g\in\mathscr{M}} \int_M |K_1^{\perp}|^2 dV_g$$

we use Proposition 3 in [10] (see also Lemma 2.6 in [22]) to conclude that

$$4\pi^2\chi(M) \geq \frac{1}{144}|Y_1^{\perp}(M)|^2.$$

In particular, the equality holds if and only if M^4 is either flat or $\mathbb{H}_c^2 \times \mathbb{H}_c^2/\Gamma$, which establishes the fist assertion.

Next, the proof look likes that one of the previous assertion. Indeed, without loss of generality we may assume that M^4 is self-dual. Hence we have

$$8\pi^2\chi(M) \geq \frac{1}{24}\int_M \left[(6w_1^+)^2 + s^2\right] dV_g$$

$$\begin{aligned} &\geq \frac{1}{48}\int_M \Big[6w_1^+ + s\Big]^2 dV_g \\ (3.11) \qquad &\geq 3\inf_{g\in\mathscr{M}}\int_M |K_1^{\perp}|^2 dV_g. \end{aligned}$$

In order to conclude it suffices to use once more Proposition 3 in [10] to arrive at

$$\chi(M) \geq \frac{1}{384\pi^2}|Y_1^{\perp}(M)|^2,$$

which gives the requested result. □

3.6. **Proof of Corollary 1.**

Proof. First, we consider $M = \mathbb{H}_c^2\times\mathbb{H}_c^2/\Gamma\sharp j(\mathbb{S}^1\times\mathbb{S}^3)$ and $e=\chi(\mathbb{H}_c^2\times\mathbb{H}_c^2/\Gamma)$. With these notations we have $\chi(M)=e-2j$ and $\tau(\mathbb{H}_c^2\times\mathbb{H}_c^2/\Gamma)=0$. Further, since $\mathbb{H}_c^2\times\mathbb{H}_c^2/\Gamma$ is the minimal model of M^4 we may invoke Theorem 3.9 of [28] to infer

$$Y(\mathbb{H}_c^2\times\mathbb{H}_c^2/\Gamma) = -8\pi\sqrt{e}.$$

We then use Proposition 3 in [39] to deduce $Y(M)=Y(\mathbb{H}_c^2\times\mathbb{H}_c^2/\Gamma)=-8\pi\sqrt{e}$.

We now suppose that M^4 admits an Einstein metric. So, since $Y_1^{\perp}(M)\leq Y(M)<0$, we can apply Theorem 1.8 to deduce

$$576\pi^2\chi(M)\geq |Y_1^{\perp}(M)|^2\geq |Y(M)|^2.$$

This tell us that

$$576\pi^2(e-2j)\geq 64e\pi^2,$$

so that $j\leq\frac{4}{9}e$, which is a contradiction. This finishes the proof of the corollary. □

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